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LETTER TO THE EDITOR

Critical behaviour of directed self-avoiding walks

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Abstract. A new directed self-avoiding walk (directed SAW) is proposed and studied on a square lattice. The critical behaviour of such directed SAWs (in two dimensions) is established and shown to be of different universality class from that of an ordinary SAW.

The SAWs on lattices, describing the excluded volume effect in linear polymers, have been studied for a long time and the critical behaviour of their statistics is well established (see e.g. de Gennes 1979). Recently, the lattice disorder has been seen to affect the pure SAW critical behaviour (Chakrabarti and Kertesz 1981, Derrida 1982, Roy and Chakrabarti 1982, and references therein). Here, we see that the specification of a direction of walk also affects the critical behaviour and the statistics of such directed SAWs have an entirely different critical behaviour (from that of an ordinary SAW).

In the percolation problem, a new critical behaviour is observed when one imposes an added restriction on the 'direction of flow' through the occupied bonds (or sites), and such a directed or oriented percolation problem is currently being studied with great interest (see e.g. Blease 1977a, b, Obukhov 1980, Dhar and Barma 1981). This difference in the critical behaviour of directed percolation from that of ordinary percolation indicates that a similar restriction on the 'direction of walk' might also affect the ordinary SAW critical behaviour. This is because one can picture the cluster geometry for ordinary percolation at the threshold (backbone cluster) as one-dimensional channels or 'links' of SAW character, having intersections or 'nodes' (Skal and Shklovskii 1975, de Gennes 1976). Although such a fractal picture is not entirely correct, it served a very useful purpose in providing a nice intuitive understanding and helped in getting some important scaling relations for ordinary percolation conduction etc (see e.g. Ziman 1979). A similar picture of the backbone cluster for directed percolation would involve 'links' which are directed SAW in nature, where a walk opposite to the specified direction is not permitted (see figure 1). This would then indicate that the critical behaviour of a directed SAW is different from that of an ordinary SAW.

We have studied here the statistics of such directed SAWs on a square lattice. For such problems, one would like to know the distribution of the number of walks $G_N(r)$ of N steps with the end-to-end distance r. For directed SAWs, however, one can easily construct a recursion relation for the total number of walks G_N for N steps. If, out of the G_{N-1} number of walks for $N-1$ steps, $(G_{N-1})_h$ are those which end up with the last $(N-1)$th step walk in the horizontal direction and $(G_{N-1})_v$ are those which

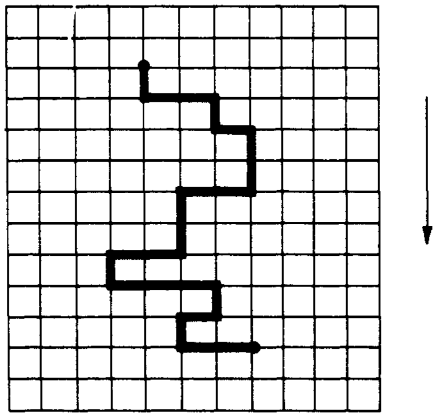

Figure 1. A part of the infinite square lattice containing a finite step directed SAW. A walk opposite to the direction specified is not permitted.

end up with the last step in the vertical (downward) direction (restriction in the upward direction—figure 1), then

$$G_N = 2(G_{N-1})_{\mathrm{h}} + 3(G_{N-1})_{\mathrm{v}} = 2G_{N-1} + (G_{N-1})_{\mathrm{v}}. \tag{1}$$

Now, since each horizontal and vertical step (walk) can generate one vertical walk for the next step,

$$(G_{N-1})_{\mathrm{v}} = G_{N-2}, \tag{2}$$

giving

$$G_N = 2G_{N-1} + G_{N-2}. \tag{3}$$

This total number of walks G_N for various steps N and the ratios G_N/G_{N-1} are listed in table 1 (up to $N = 14$).

Table 1.

Number of steps (N)	Total number of walks (G_N)	Ratio (G_N/G_{N-1})	Average end-to-end distance ($\bar{R}_N$)
1	3	—	1.0000
2	7	2.3333	1.6653
3	17	2.4286	2.2255
4	41	2.4118	2.7852
5	99	2.4146	3.3220
6	239	2.4141	3.8510
7	577	2.4142	4.3728
8	1 393	2.4142	4.8903
9	3 363	2.4142	5.4045
10	8 119	2.4142	5.9161
11	19 601	2.4142	6.4245
12	47 321	2.4142	6.9303
13	114 243	2.4142	7.4232
14	275 807	2.4142	7.9022

As one can clearly see from the table (in fact, checking these G_N values for steps up to $N = 100$, we have seen this), expressing G_N as (see e.g. McKenzie 1976)

$$G_N = \mu^N N^{\gamma-1}, \tag{4}$$

for $N \to \infty$, one gets

$$\mu = 2.4142, \tag{5}$$

$$\gamma = 1, \tag{6}$$

for a directed SAW on a square lattice, compared to $\mu = 2.6385$ and $\gamma = \frac{4}{3}$ for ordinary SAWs on a square lattice (McKenzie 1976). In fact, (5) and (6) follow directly from the recursion relation (3); with $G_N \sim \mu^N$, the relation (3) reduces to a quadratic equation in μ giving $\mu = 1 + \sqrt{2}$.

To calculate the average end-to-end distance $\bar{R}_N$ for N steps (first moment of the distribution), we simulated such walks on an Uptron S800 desk computer, following the algorithm (modifying for such directed SAWs) for exact enumeration of ordinary SAWs, outlined by Martin (1974). The results (up to $N = 14$) are also given in table 1. Expressing $\bar{R}_N$, for $N \to \infty$, as

$$\bar{R}_N \sim N^{\nu}, \tag{7}$$

we find (see figure 2) that

$$\nu = 0.86 \pm 0.02 \tag{8}$$

for a directed SAW in two dimensions, compared with $\nu = 0.75$ for an ordinary SAW (McKenzie 1976). It may be mentioned here that, although with the desk computer at hand, we could not go beyond $N = 14$ for $\bar{R}_N$ calculations, we believe the value of ν obtained here is the correct asymptotic ($N \to \infty$) limit value, because of the very accurate fitting of the above ν value for $N = 10$ to 14 (see figure 2) and the observed saturation of G_N/G_{N-1} ratios to the asymptotic μ value beyond $N = 6$ (see table 1).

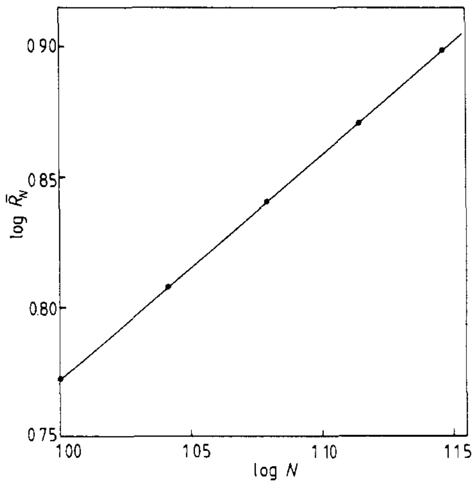

Figure 2. Plot of log $\bar{R}_N$ against log N. The slope of the straight line fit gives the exponent ν for the directed SAW in two dimensions.

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